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SYSTEMS OF PLANE CURVES WHOSE INTRINSIC EQUATIONS ARE ANALOGOUS TO THE INTRINSIC EQUATION OF AN ISOTHERMAL SYSTEM.*

By H. W. Reddick.

If we take the differential equation of a singly infinite system of plane curves in the form

$$y' = \tan \lambda(x, y), \tag{1}$$

the curvature κ of a curve of the system and the curvature κ_1 of a curve of the orthogonal system are given by the formulas:

$$\kappa = \lambda_x \cos \lambda + \lambda_y \sin \lambda, \quad \kappa_1 = \lambda_y \cos \lambda - \lambda_x \sin \lambda, \tag{2}$$

where the subscripts x and y denote partial differentiation.

Connected with this system of curves and its orthogonal system are the four intrinsic quantities T, N, T_1 and N_1, defined as follows:

$T = \dfrac{d\kappa}{ds}$ = the rate of variation of the curvature of a curve of the given system (1) along the curve itself in the direction taken arbitrarily as positive.

$N = \dfrac{d\kappa}{dn}$ = the rate of variation of the curvature of a curve of the given system along a curve of the orthogonal system in the positive direction, which is taken to be the direction making an angle of + 90° with the positive direction of the curve of the given system.

$T_1 = \dfrac{d\kappa_1}{ds_1}$ = the rate of variation of the curvature of a curve of the orthogonal system along the curve itself in the positive direction.

$N_1 = \dfrac{d\kappa_1}{dn_1}$ = the rate of variation of the curvature of a curve of the orthogonal system along a curve of the given system in the direction making an angle of + 90° with the positive direction of the curve of the orthogonal system.

It is known that $T + T_1 = 0$† is the intrinsic equation of the important isothermal system. It would seem that equations analogous, from this point of view, to the intrinsic equation of an isothermal system (in particular, the equation $T - T_1 = 0$) might also represent interesting systems. It is the purpose of this paper to consider the twelve systems of curves whose intrinsic equations are formed by equating to zero the sums and differences of the four quantities T, N, T_1 and N_1, taken in pairs.

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† See Kasner, "The Riccati Differential Equations which Represent Isothermal Systems," Bull. Am. Math. Soc., vol. 10, p. 342.

The Equations of the Twelve Systems.

We now enumerate the twelve systems as given by their intrinsic equations:

$$\begin{array}{rlrl}
T + T_1 = 0, & \text{(I)} & N + T_1 = 0, & \text{(VII)} \\
N + N_1 = 0, & \text{(II)} & N_1 - T = 0, & \text{(VIII)} \\
T - T_1 = 0, & \text{(III)} & N_1 + T_1 = 0, & \text{(IX)} \\
N - N_1 = 0, & \text{(IV)} & N_1 - T_1 = 0, & \text{(X)} \\
N + T = 0, & \text{(V)} & N_1 + T = 0, & \text{(XI)} \\
N - T = 0, & \text{(VI)} & N - T_1 = 0. & \text{(XII)}
\end{array}$$

If we change T into T_1 and N into N_1, equations I, II, III and IV remain unaltered, which shows that each of these systems has the same intrinsic equation as its orthogonal system, but equations V, VI, VII and VIII are changed into IX, X, XI and XII respectively. Systems IX, X, XI and XII are therefore orthogonal to systems V, VI, VII and VIII respectively. It will be sufficient, then, to deal with the first eight systems.

The values of T, N, T_1 and N_1 in terms of λ and its partial derivatives are:

$$\begin{aligned}
T &= [\alpha - (\gamma - \delta) \tan \lambda + \beta \tan^2 \lambda] \cos^2 \lambda, \\
N &= [\delta - (\alpha - \beta) \tan \lambda + \gamma \tan^2 \lambda] \cos^2 \lambda, \\
T_1 &= [\beta + (\gamma - \delta) \tan \lambda + \alpha \tan^2 \lambda] \cos^2 \lambda, \\
N_1 &= [\gamma + (\alpha - \beta) \tan \lambda + \delta \tan^2 \lambda] \cos^2 \lambda,
\end{aligned} \tag{3}$$

where

$$\alpha = \lambda_{xx} + \lambda_x\lambda_y, \qquad \gamma = \lambda_x^2 - \lambda_{xy},$$
$$\beta = \lambda_{yy} - \lambda_x\lambda_y, \qquad \delta = \lambda_y^2 + \lambda_{xy}.$$

The equations of systems I–VIII may then be written in the form:

$$\alpha + \beta = \lambda_{xx} + \lambda_{yy} = 0, \tag{I$'$}$$
$$\gamma + \delta = \lambda_x^2 + \lambda_y^2 = 0, \tag{II$'$}$$
$$\alpha - \beta - (\gamma - \delta) \tan 2\lambda = 0, \tag{III$'$}$$
$$\alpha - \beta + (\gamma - \delta) \cot 2\lambda = 0, \tag{IV$'$}$$

$$\alpha + \delta - (\alpha - \beta + \gamma - \delta) \tan \lambda + (\beta + \gamma) \tan^2 \lambda = 0, \qquad (\mathrm{V}')$$

$$\alpha - \delta + (\alpha - \beta - \gamma + \delta) \tan \lambda + (\beta - \gamma) \tan^2 \lambda = 0, \qquad (\mathrm{VI}')$$

$$\beta + \delta - (\alpha - \beta - \gamma + \delta) \tan \lambda + (\alpha + \gamma) \tan^2 \lambda = 0, \qquad (\mathrm{VII}')$$

$$\alpha - \gamma - (\alpha - \beta + \gamma - \delta) \tan \lambda + (\beta - \delta) \tan^2 \lambda = 0. \qquad (\mathrm{VIII}')$$

The equations of systems IX–XII could be written down from those of V–VIII by changing $\tan \lambda$ into $- \cot \lambda$.

System I is the familiar isothermal system. The equation $y' = \tan \lambda (x, y)$, where λ is a solution of Laplace's Equation I′, represents all isothermal families of plane curves.*

System II is composed of all families of parallel straight lines in the plane, since we have from equation II′, $\lambda_x = \lambda_y = 0$, hence $\lambda = \text{const.}$, and the equation of the system is $y' = \text{const.}$, the ∞^1 families of parallel straight lines in the plane.

System IV is the 45° isogonal system of III, since equation IV′ becomes identical with equation III′ if λ is replaced by $(\pi/4) + \lambda$.

It remains to consider the five systems III, V, VI, VII and VIII. The general solution will not be found in these cases, but in each case a family of ∞^3 curves belonging to the system will be found. This family of ∞^3 curves will be composed of ∞^2 subfamilies of ∞^1 curves each, such that all the curves of any subfamily may be obtained from any one of them by translation. That is, we shall find all systems of types III, V, VI, VII and VIII admitting a group of translations. We shall also find in the last section a characteristic geometric property of the complete system III.

We notice that $\lambda = \text{const.}$ satisfies the equation of each system, hence $y' = \text{const.}$ is a particular solution in each case. This solution will be disregarded in the following discussion.

We also note one obvious particular solution of equation III′, namely $T = T_1 = 0$. The equation $T = 0$ represents all circles in the plane (including straight lines) since a circle is the only real plane curve whose curvature is constant. The equation $T_1 = 0$ represents a system of curves whose orthogonal curves are circles. Hence $T = T_1 = 0$ represents all systems of circles whose orthogonal curves are also circles. Such systems are also isothermal, since $T = T_1 = 0$ is a solution of equation I′.

Systems of Type III Admitting a Group of Translations.

Systems admitting a group of translations will be of the form

$$y' = \tan \lambda(x)$$

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* Lie-Scheffers, Differentialgleichungen, p. 157.

if the y-axis is taken as the direction of translation. To obtain such systems we make λ a function of x alone in equation III′. This gives

$$\lambda'' - \lambda'^2 \tan 2\lambda = 0, \tag{4}$$

the primes denoting derivatives with respect to x. The first integral of (4) is easily found to be

$$\lambda' \sqrt{\cos 2\lambda} = c_1. \tag{5}$$

The integration of equation (5) involves elliptic integrals. Writing it in the form

$$c_1 dx = \sqrt{1 - 2\sin^2 \lambda}\, d\lambda, \tag{6}$$

and making the substitution $\sqrt{2}\sin\lambda = \sin\phi$, we have

$$c_1 x = \frac{1}{\sqrt{2}} \int \frac{\cos^2 \phi d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} + c_2. \tag{7}$$

We now apply the formula

$$\int_0^\phi \frac{\cos^2 \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{1}{k^2}[E(k, \phi) - k'^2 F(k, \phi)], \tag{8}$$

where $k^2 < 1$, $k'^2 = 1 - k^2$ and F and E are elliptic integrals of the first and second classes:

$$F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}; \qquad E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\, d\phi.$$

Writing $\sqrt{2}c_1 = a$ and $\sqrt{2}c_2 = b$, equation (7) becomes

$$ax = 2E\left(\frac{1}{\sqrt{2}}, \phi\right) - F\left(\frac{1}{\sqrt{2}}, \phi\right) + b.$$

We obtain y from the relation

$$dy = \tan\lambda\, dx.$$

Hence, from (6),

$$c_1 dy = \tan\lambda \sqrt{1 - 2\sin^2\lambda}\, d\lambda.$$

Making the same substitution as before, $\sqrt{2}\sin\lambda = \sin\phi$, and integrating, we obtain

$$c_1 y = \tan^{-1}\cos\phi - \cos\phi + c_3.$$

Writing as before $\sqrt{2}c_1 = a$, also $\sqrt{2}c_3 = c$, this becomes

$$ay = \sqrt{2}(\tan^{-1}\cos\phi - \cos\phi) + c.$$

We have, then, the solution of the system in parametric form:*

$$
\begin{aligned}
ax &= 2E\left(\frac{1}{\sqrt{2}}, \phi\right) - F\left(\frac{1}{\sqrt{2}}, \phi\right) + b, \\
ay &= \sqrt{2}\,(\tan^{-1}\cos\phi - \cos\phi) + c,
\end{aligned}
\tag{9}
$$

where a, b and c are arbitrary constants and ϕ is the parameter. For given values of a and b we have a family of ∞^1 curves admitting a group of translations along the y-axis. The variation of a and b gives ∞^2 such families.

As all of these curves may be obtained from any one of them by a translation and a homothetic transformation, there is essentially only one curve, which may be obtained, for example, by letting $a = 1$, $b = c = 0$, in equations (9)

The curve is symmetrical with respect to the y-axis and is periodic. It crosses the x-axis at an angle of 45° and is an elongated analogue of the sine curve.

Systems of Types V and VI Admitting a Group of Translations.

The substitution of $\lambda = \lambda(x)$ in equations V′ and VI′ leads to the same differential equation in both cases, namely

$$\lambda'' - \lambda'^2 \tan\lambda = 0 \tag{10}$$

or

$$y'''(1 + y'^2) = 3y'y''^2, \tag{11}$$

which is the differential equation of all circles in the plane. This system of ∞^3 circles is composed of ∞^2 families of the form $y' = \tan\lambda(x)$, each admitting a group of translations along the y-axis.

For any system of circles, $T = 0$, since the variation of the curvature of a circle along the circle is zero. In a family of ∞^1 circles obtained from a given circle by translation, $N = 0$, since the variation of the curvature of a circle of the family along an orthogonal curve is zero. The only solution of either $N + T = 0$ or $N - T = 0$ of the form $y' = \tan\lambda(x)$ is the simultaneous solution for which $T = N = 0$. This represents the totality of families of ∞^1 circles obtained by translating a circle in the direction of the y-axis.

Systems of Types VII and VIII Admitting a Group of Translations.

The substitution of $\lambda = \lambda(x)$ in equations VII′ and VIII′ gives in both cases

$$\lambda'^2 - \lambda'' + (\lambda'^2 + \lambda'')\tan\lambda = 0. \tag{12}$$

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* An isothermal system (other than parallel straight lines) which admits a group of translations is of the form $y + \log\cos x = c$. See Kasner, loc. cit., p. 342.

The first integral of this equation is easily found to be

$$\lambda'(\cos\lambda - \sin\lambda) = c_1. \tag{13}$$

Integrating again, we find

$$c_1 x = \sin\lambda + \cos\lambda + c_2.$$

Also, from (13) and the relation

$$dy = \tan\lambda \, dx,$$

we find,

$$c_1 y = \sin\lambda - \cos\lambda - \log(\sec\lambda + \tan\lambda) + c_3.$$

Hence we have the equations of the system in parametric form:

$$\begin{aligned} ax &= \sin\lambda + \cos\lambda + b, \\ ay &= \sin\lambda - \cos\lambda - \log(\sec\lambda + \tan\lambda) + c. \end{aligned} \tag{14}$$

A Characteristic Geometric Property of the Complete System $T - T_1 = 0$.

The vanishing of the function

$$T = \frac{d\kappa}{ds} = \frac{\lambda_{xx} + \lambda_x\lambda_y - (\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy})y' + (\lambda_{yy} - \lambda_x\lambda_y)y'^2}{1 + y'^2} \tag{15}$$

at any point (x, y) of a curve of the system

$$y' = \tan\lambda(x, y) \tag{16}$$

is the condition that the curve is hyperosculated at this point by its circle of curvature, since the rate of variation of the curvature of the curve at this point is the same as that of a circle, namely zero.

Let us now consider a system of curves (16) and the system of ∞^1 isogonals through any point (x, y). The slope of an isogonal making an angle α with a curve of the given system (16) is

$$y' = \tan(\lambda + \alpha), \tag{17}$$

and the function T for this isogonal is of the same form (15) as for a curve of system (16), where y' now has the value given in equation (17). As α varies, equation (15) gives the value of T at the given point for each of the ∞^1 isogonals through this point. The function T will vanish for two of these isogonals whose directions are given by the roots of the equation

$$\lambda_{xx} + \lambda_x\lambda_y - (\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy})y' + (\lambda_{yy} - \lambda_x\lambda_y)y'^2 = 0, \tag{18}$$

regarded as a quadratic in y', the coefficients being determined for the given point (x, y). The isogonals having these two directions are hyperosculated at the given point by their circles of curvature.

The condition that these two directions be perpendicular is

$$\lambda_{xx} + \lambda_{yy} = 0, \quad \text{or} \quad T + T_1 = 0,$$

which is the equation of an isothermal system. This is the characteristic geometric property of isothermal systems found by Kasner.*

We shall now obtain a characteristic property of the system, $T - T_1 = 0$, by considering its isogonal system. We wish to find the relation between the roots of equation (18) subject to the condition

$$T - T_1 = 0$$

or

$$\lambda_{xx} - \lambda_{yy} + 2\lambda_x\lambda_y - (\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy}) \tan 2\lambda = 0. \tag{19}$$

Representing the roots of equation (18) by $\tan \theta_1$ and $\tan \theta_2$, we find

$$\frac{\tan \theta_1 \tan \theta_2 - 1}{\tan \theta_1 + \tan \theta_2} = \frac{\lambda_{xx} - \lambda_{yy} + 2\lambda_x\lambda_y}{\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy}},$$

which, under condition (19), becomes

$$-\cot(\theta_1 + \theta_2) = \tan 2\lambda.$$

Hence the directions $\theta_1 + \theta_2$ and 2λ are orthogonal. Also, if the directions $\theta_1 + \theta_2$ and 2λ are orthogonal, the system is of the form (19). We have, then, a characteristic property of a system of curves of the form $T - T_1 = 0$:

The family of ∞^1 *curves through a given point isogonal to a given simply infinite system,* $y' = \tan \lambda(x, y)$, *contains two isogonals which are hyperosculated by their circles of curvature. The directions* θ_1 *and* θ_2 *of these two isogonals are such that the direction* $\theta_1 + \theta_2$ *is orthogonal to the direction* 2λ *for every point of the plane when, and only when, the given system is of the form* $T - T_1 = 0$.

COLUMBIA UNIVERSITY,
October, 1912.

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[*] Bull. Am. Math. Soc., vol. 14, p. 170.